The asymptotic values of the general Zagreb and Randić indices of trees with bounded maximum degree[*]

Xueliang Li, Yiyang Li

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, China

Abstract

Let $\mathcal{T}_n^\Delta$ denote the set of trees of order n, in which the degree of each vertex is bounded by some integer Δ. Suppose that every tree in $\mathcal{T}_n^\Delta$ is equally likely. We show that the number of vertices of degree j in $\mathcal{T}_n^\Delta$ is asymptotically normal with mean $(\mu_j + o(1))n$ and variance $(\sigma_j + o(1))n$, where μ_j, σ_j are some constants. As a consequence, we give estimate to the value of the general Zagreb index for almost all trees in $\mathcal{T}_n^\Delta$. Moreover, we obtain that the number of edges of type (i, j) in $\mathcal{T}_n^\Delta$ also has mean $(\mu_{ij} + o(1))n$ and variance $(\sigma_{ij} + o(1))n$, where an edge of type (i, j) means that the edge has one end of degree i and the other of degree j, and μ_{ij}, σ_{ij} are some constants. Then, we give estimate to the value of the general Randić index for almost all trees in $\mathcal{T}_n^\Delta$.

Keywords: generating function, tree, normal distribution, asymptotic value, general Zagreb index, general Randić index.

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1 Introduction

In this paper, we mainly consider trees, in which the degree of each vertex is bounded by some integer Δ. If $\Delta = 1, 2$, the cases are trivial. Thus, we suppose $\Delta \geq 3$ throughout this paper. Let $\mathcal{T}_n^\Delta$ denote the set of trees with n vertices. We suppose that every tree in $\mathcal{T}_n^\Delta$ is equally likely and X_n is a random variable, such as the number of vertices of degree j, or the number of edges of type (i, j), each having one end of degree i and the other of degree

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j. It is easy to see that X_n can take at most $|\mathcal{T}_n^\Delta|$ distinct values. We first introduce two generating functions. Setting $t_n = |\mathcal{T}_n^\Delta|$, we have

$$t(x) = \sum_{n \geq 1} t_n x^n,$$

$$t(x, u) = \sum_{n \geq 1, k \geq 0} t_{n,k} x^n u^k,$$

where $t_{n,k}$ denotes the number of trees in $\mathcal{T}_n^\Delta$ such that $X_n = k$. Therefore, the probability of X_n can be defined as

$$\Pr[X_n = k] = \frac{t_{n,k}}{t_n}.$$

Note that $t(x, 1) = t(x)$. In [11], it is showed that t_n is asymptotically equal to $\tau \cdot \frac{x_0^{-n}}{n^{5/2}}$, where τ and x_0 are constants with $x_0 \leq 1/2$.

In conjunction with the generating functions and asymptotic analysis, in [4] and [13] the authors investigated the limiting distribution of the number of vertices of given degree j for trees without degree restriction. By the same method, many results have been established for other variables, such as the number of a given *path* or *pattern* (see [8]) for rooted trees, planar trees, labeled trees *et al.* However, all the statements showed that the limiting distributions are normal. We refer the readers to [2] and [8] for further details.

In this sequel, we follow the method used in [2] and [4] to obtain that the distribution of the number of vertices of degree j for trees in $\mathcal{T}_n^\Delta$ is also asymptotically normal with mean $(\mu_j + o(1))n$ and variance $(\sigma_j + o(1))n$. Then, we give estimate to the value of the general Zagreb index for almost all trees in $\mathcal{T}_n^\Delta$. However, for the number of edges of type (i, j), we only get a weak statement which can not show that the limiting distribution is normal. Nevertheless, we still can use it to obtain the asymptotical value of the general Randić index for almost all trees in $\mathcal{T}_n^\Delta$.

The definitions of the general Zagreb index and general Randić index will be given in next sections. Many results have been obtained for the two parameters. We refer the readers to [9] and [10] for a detailed survey. In this paper we will show that for the random space $\mathcal{T}_n^\Delta$, each of the indices has a value of $\Theta(n)$ for almost all trees.

Section 2 is devoted to a systematic treatment of the number of vertices of degree j and the general Zagreb index. In Section 3, we investigate the number of edges of type (i, j) and the general Randić index.

2 The number of vertices of degree j

In this section, we first consider the the limiting distribution of the number of vertices of degree j in $\mathcal{T}_n^\Delta$. Then, as an immediate consequence, we get the asymptotic value of the general Zagreb index for almost all trees in $\mathcal{T}_n^\Delta$.

In what follows, we introduce some terminology and notation which will be used in the sequel. For the others not defined here, we refer to book [7].

Analogous to trees, we introduce generating functions for rooted trees and planted trees. Let $\mathcal{R}_n^\Delta$ denote the set of rooted trees of order n with degrees bounded by an integer Δ. Setting $r_n = |\mathcal{R}_n^\Delta|$, we have

$$r(x) = \sum_{n \geq 1} r_n x^n$$

and

$$r(x, u) = \sum_{n \geq 1, k \geq 0} r_{n,k} x^n u^k,$$

where $r_{n,k}$ denotes the number of trees in $\mathcal{R}_n^\Delta$ such that X_n equals k. A *planted tree* is formed by adding a vertex to the root of a rooted tree. The new vertex is called the *plant*, and we never count it in the sequel. Analogously, let $\mathcal{P}_n^\Delta$ denote the set of planted trees with n vertices of bounded maximum degree Δ. Setting $p_n = |\mathcal{P}_n^\Delta|$, we have

$$p(x) = \sum_{n \geq 1} p_n x^n$$

and

$$p(x, u) = \sum_{n \geq 1, k \geq 0} p_{n,k} x^n u^k,$$

where $p_{n,k}$ denotes the number of trees in $\mathcal{P}_n^\Delta$ such that X_n equals k. By the definition of planted trees, one can readily see that $p(x, 1) = p(x) = r(x, 1) = r(x)$.

Furthermore, we introduce another generating function $p^{(\Delta-1)}(x)$. Denote $p_n^{(\Delta-1)}$ as the number of planted trees such that the degree of the root is bounded by $\Delta - 1$, while the degrees of other vertices are bounded by Δ. Then, we define

$$p^{(\Delta-1)}(x) = \sum_{n \geq 1} p_n^{(\Delta-1)} x^n.$$

In [11], Otter showed that there exists a number x_0 such that

$$p(x) = b_1 + b_2 \sqrt{x_0 - x} + b_3(x_0 - x) + \cdots, \tag{1}$$

where b_1, b_2, b_3 are some constants not equal to zero. Evidently, $p(x_0) = b_1$ and for any $|x| \leq x_0$, $p(x)$ is convergent. For any Δ, $x_0 \leq 1/2$; particularly, if $\Delta = 4$, $x_0 \approx 0.3551817$ and $p(x_0) \approx 1.117421$. Moreover,

$$p^{(\Delta-1)}(x_0) = 1. \tag{2}$$

We refer the readers to [11] for more details.

The proofs of our main results in this paper ultimately rely on the following lemma, due to Chyzak *et al.* [2] and Drmota [3]. We first introduce some notation.

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Let $\mathbf{y}(x, u) = (y_1(x, u), \ldots, y_N(x, u))^T$ be a column vector. We suppose that $G(x, \mathbf{y}, u)$ is an analytic function with non-negative integer Taylor coefficients. $G(x, \mathbf{y}, u)$ can be expanded as

$$G(x, \mathbf{y}, u) = \sum_{n \geq 1, k \geq 0} g_{n,k} x^n u^k.$$

Let X_n denote a random variable with probability

$$\Pr[X_n = k] = \frac{g_{n,k}}{g_n}, \tag{3}$$

where $g_n = \sum_k g_{n,k}$.

Lemma 1. *Let $\mathbf{F}(x, \mathbf{y}, u) = (F_1(x, \mathbf{y}, u), \ldots, F_N(x, \mathbf{y}, u))^T$ be functions analytic around $x = 0$, $\mathbf{y} = (y_1, \ldots, y_N)^T = \mathbf{0}$, $u = 0$, with Taylor coefficients all are non-negative integers. Suppose $\mathbf{F}(0, \mathbf{y}, u) = \mathbf{0}$, $\mathbf{F}(x, \mathbf{0}, u) \neq \mathbf{0}$, $\mathbf{F}_x(x, \mathbf{y}, u) \neq \mathbf{0}$, and for some j, $\mathbf{F}_{y_j y_j}(x, \mathbf{y}, u) \neq \mathbf{0}$. Furthermore, assume that $x = x_0$ together with $\mathbf{y} = \mathbf{y}_0$ is a non-negative solution of the system of equations*

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, 1) \tag{4}$$

$$0 = det(\mathbf{I} - \mathbf{F}_y(x, \mathbf{y}, 1)) \tag{5}$$

inside the region of convergence of $\mathbf{F}$, $\mathbf{I}$ is the unit matrix. Let $\mathbf{y} = (y_1(x, u), \ldots, y_N(x, u))^T$ denote the analytic solution of the system

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, u) \tag{6}$$

with $\mathbf{y}(0, u) = \mathbf{0}$.

If the dependency graph $G_{\mathbf{F}}$ of the function system Equ.(6) is strongly connected, then there exist functions $f(u)$ and $g_i(x, u)$, $h_i(x, u)$ $(1 \leq i \leq N)$ which are analytic around $x = x_0$, $u = 1$, such that

$$y_i(x, u) = g_i(x, u) - h_i(x, u) \sqrt{1 - \frac{x}{f(u)}} \tag{7}$$

is analytically continued around $u = 1$, $x = f(u)$ with $arg(x - f(u)) \neq 0$, where $x = f(u)$ together with $y = y(f(u), u)$ is the solution of the extended system

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}, u) \tag{8}$$

$$0 = det(\mathbf{I} - \mathbf{F}_y(x, \mathbf{y}, u)). \tag{9}$$

Moreover, let $G(x, \mathbf{y}, u)$ be an analytic function with non-negative Taylor coefficients such that the point $(x_0, \mathbf{y}(x_0, 1), 1)$ is contained in the region of convergence. Finally, let X_n be the random variable defined in Equ.(3). Then the random variable X_n is asymptotically normal with mean

$$E(X_n) = \mu n + O(1) \ (n \to \infty),$$

and variance

$$Var(X_n) = \sigma n + O(1) \ (n \to \infty)$$

with $\mu = \frac{-f'(1)}{f(1)}$.

Remark 1: We say that the *dependency graph* $G_{\mathbf{F}}$ of $\mathbf{y} = \mathbf{F}(x, \mathbf{y}, u)$ is strongly connected if there is no subsystem of equations that can be solved independently from others. If $G_{\mathbf{F}}$ is strongly connected, then $\mathbf{I} - \mathbf{F_y}(x_0, \mathbf{y_0}, 1)$ has rank $N - 1$. Suppose that $\mathbf{v}^T$ is a vector with $\mathbf{v}^T(\mathbf{I} - \mathbf{F_y}(x_0, \mathbf{y_0}, 1)) = 0$. Then, $\mu = \frac{\mathbf{v}^T(\mathbf{F}_u(x_0, \mathbf{y_0}, 1))}{x_0 \mathbf{v}^T(\mathbf{F}_x(x_0, \mathbf{y_0}, 1))}$. We refer the readers to [2, 3] for more details.

In what follows, we shall use the above lemma to investigate the number of vertices of degree j in T_n^{Δ}, where j is a given integer.

Firstly, we focus on the planted trees. There appears an expression of the form $Z(S_n, f(x, u))$ (or $f(x)$), which is the substitution of the counting series $f(x, u)$ (or $f(x)$) into the cycle index $Z(S_n)$ of the symmetric group S_n. This involves replacing each variable s_i in $Z(S_n)$ by $f(x^i, u^i)$ (or $f(x^i)$). For instance, if $n = 3$, then $Z(S_3) = (1/3!)(s_1^3 + 3s_1s_2 + 2s_3)$, and $Z(S_3, f(x, u)) = (1/3!)(f(x, u)^3 + 3f(x, u)f(x^2, u^2) + 2f(x^3, u^3))$. We refer the readers to [7] for details.

Note that a planted tree with a root of degree k can be viewed as a root vertex attached by $k-1$ planted trees. Employing the classic Pólya enumeration theorem, we have $Z(S_{k-1}; p(x))$ as the counting series of the planted trees whose roots have degree k, and the coefficient of x^p in $x \cdot Z(S_{k-1}; p(x))$ is the number of planted trees with p vertices (see [7] p.51–54). Therefore,

$$p(x) = x \cdot \sum_{k=0}^{\Delta-1} Z(S_k; p(x)),$$

and

$$p^{(\Delta-1)}(x) = x \cdot \sum_{k=0}^{\Delta-2} Z(S_k; p(x)).$$

By the same method, we can obtain that

$$p(x, u) = x \cdot \sum_{l=1}^{\Delta} Z(S_{l-1}; p(x, u)) + x(u - 1)Z(S_{j-1}; p(x, u)), \tag{10}$$

where the last term $(xu - x)Z(S_{j-1}; p(x, u))$ serves to count the vertices of degree j when the root of a planted tree is of degree j. Then, we show that this equation satisfies the conditions of Lemma 1. Suppose $p(x, u) = F(x, p(x, u), u)$. It is well-known that the partial derivative of $Z(S_n; \cdot)$ enjoys (see [4])

$$\frac{\partial}{\partial s_1} Z(S_n; s_1, \ldots, s_n) = Z(S_{n-1}; s_1, \ldots, s_{n-1}). \tag{11}$$

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One can readily see that

$$F_p(x_0, p(x_0, 1), 1) = x_0 \sum_{k=0}^{\Delta-2} Z(S_k; p(x_0, 1)) = p^{(\Delta-1)}(x_0) = 1.$$

The other conditions are easy to be illustrated. Thus we have that $p(x, u)$ is in the form of

$$p(x, u) = g_1(x, u) - h_1(x, u)\sqrt{1 - \frac{x}{f(u)}}, \tag{12}$$

where $g_1(x, u)$, $h_1(x, u)$ and $f(u)$ are analytic around $x = x_0$ and $u = 1$, and $p(x, u)$ is analytically continued around $u = 1$, $x = f(u)$ with $\arg(x - f(u)) \neq 0$. From Equ.(1), we can see $f(1) = x_0$.

Analogous to Equ.(10), for rooted trees, it follows that

$$r(x, u) = x \cdot \sum_{l \geq 0}^{\Delta} Z(S_l; p(x, u)) + (xu - x)Z(S_j; p(x, u)). \tag{13}$$

For trees, however, in [11] the author obtained that

$$t(x) = r(x) - \frac{1}{2}p(x)^2 + \frac{1}{2}p(x^2), \tag{14}$$

and $t(x)$ can be expanded as

$$t(x) = c_1 + c_2(1 - \frac{x}{x_0}) + c_3(1 - \frac{x}{x_0})^3 + \cdots, \tag{15}$$

where c_1, c_2, c_3 are some constants not equal to zero. Furthermore, we can also obtain a similar equation for $t(x, u)$. We introduce a useful lemma due to Otter [11].

Two edges in a tree are *similar*, if they are the same under some automorphism of the tree. To *join* two planted trees is to connect the two roots with a new edge and get rid of the two plants. If the two panted trees are the same, we say that the new edge is *symmetric.*

Lemma 2. *For any tree, the number of rooted trees corresponding to this tree minus the number of nonsimilar edges (except for the symmetric edge) is the number* 1.

Note that, if we delete any one edge from a similar set in a tree, the yielded trees are the same two trees. Hence, different pairs of planted trees correspond to nonsimilar edges. Now, we have

$$t(x, u) = r(x, u) - \frac{1}{2}p(x, u)^2 + \frac{1}{2}p(x^2, u^2). \tag{16}$$

Here, we should notice that $t(x, u)$ is a function in $p(x, u)$, but its Taylor coefficient of $p(x, u)$ is not sure to be non-negative. Thus, we could not use Lemma 1 for $t(x, u)$.

However, Equ.(16) together with Equs.(12) and (13) gives

$$t(x, u) = \widetilde{g}(x, u) - \widetilde{h}(x, u)\sqrt{1 - \frac{x}{f(u)}},$$

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where $\widetilde{g}(x,u)$, $\widetilde{h}(x,u)$ are analytic around $x = f(1)$ and $u = 1$, and $t(x,u)$ is also analytically continued around $x = f(u)$ and $u = 1$ with $\arg(x - f(u)) \neq 0$.

In what follows, we shall show that $\widetilde{h}(f(u),u) = 0$ around $u = 1$. We have

$$t(x,u) = p(x,u) - x(u-1)Z(S_{j-1}; p(x,u)) + x \cdot Z(S_\Delta; p(x,u))$$
$$+ x(u-1)Z(S_j; p(x,u)) - \frac{1}{2}p(x,u)^2 + \frac{1}{2}p(x^2,u^2)$$
$$= g_1 - h_1\sqrt{1 - \frac{x}{f(u)}} - x(u-1)Z(S_{j-1}; g_1 - h_1\sqrt{1 - \frac{x}{f(u)}})$$
$$+ x \cdot Z(S_\Delta; g_1 - h_1\sqrt{1 - \frac{x}{f(u)}}) + x(u-1)Z(S_j; g_1 - h_1\sqrt{1 - \frac{x}{f(u)}})$$
$$- \frac{1}{2}(g_1^2 + h_1^2(1 - \frac{x}{f(u)}))$$
$$+ g_1 h_1 \sqrt{1 - \frac{x}{f(u)}} + p(x^2, u^2).$$

Then, by means of Taylor's theorem we get

$$Z(S_k; g_1 - h_1\sqrt{1 - \frac{x}{f(u)}}) = \sum_{i=0}^{k} Z^{(i)}(S_k; g_1)h_1^i(1 - x/f(u))^{i/2}\frac{(-1)^i}{i!},$$

where $Z^{(i)}$ denotes the i-th derivative with respect to the cycle index s_1. Thus, we obtain

$$\widetilde{h} = h_1(1 - g_1 + x \cdot Z(S_{\Delta-1}; g_1) + x(u-1)Z(S_{j-1}; g_1) - x(u-1)Z(S_{j-2}; g_1)).$$

On the other hand, from Lemma 1, note that $x = f(u)$ and $p(x,u) = g_1(f(u),u)$ are the solutions of

$$p = x \cdot \sum_{l=1}^{\Delta} Z(S_{l-1}; p) + x(u-1)Z(S_{j-1}; p),$$
$$1 = x \cdot \sum_{l=1}^{\Delta-1} Z(S_{l-1}; p) + x(u-1)Z'(S_{j-1}; p),$$

which yields

$$g_1(f(u),u) = 1 + f(u) \cdot Z(S_{\Delta-1}; g_1) + f(u)(u-1)(Z(S_{j-1}; g_1) - Z(S_{j-2}; g_1)),$$

that is,

$$\widetilde{h}(f(u),u) = 0.$$

Then by setting $h(x,u) = \widetilde{h}(x,u)(1 - \frac{x}{f(u)})^{-1}$, for some $g(x,u)$ it follows that

$$t(x,u) = g(x,u) - h(x,u)(1 - \frac{x}{f(u)})^{3/2},$$

around $x = f(u)$ and $u = 1$ with $\arg(x - f(u)) \neq 0$. Moreover, we can see $h(x, u)$ is analytic around $x = f(1)$ and $u = 1$. By Equ.(15), we have $h(f(1), 1) \neq 0$, and thus $h(f(u), u) \neq 0$ around $u = 1$.

Next, we need the following proposition from [4], which can be proved by a transfer lemma of Flajolet and Odlyzko [5] and Cauchy's formula. We refer the readers to [3, 4] and [5] for more details.

Proposition 1. *Suppose $y(x, u) = \sum y_{nm} x^n u^m$ is an analytic function with $y_{nm} \geq 0$. There exist functions $g(x, u)$, $h(x, u)$ and $f(u)$ which are analytic around $x = x_0 = f(1)$ and $u = 1$, x_0 is the radius of convergence of $y(x, 1)$, $y(x, u)$ is analytically continued in the region $|x - f(u)| < \eta$, $\arg(x - f(u)) \neq 0$ and $|u - 1| < \eta$, where η is sufficiently small, and*

$$y(x, u) = g(x, u) - h(x, u)(1 - \frac{x}{f(u)})^{3/2}.$$

Then $y_n(u) = \sum_m y_{nm} u^m$ is asymptotically given by

$$y_n(u) = \frac{3h(f(u), u)}{4\sqrt{\pi} n^{5/2}} f(u)^{-n+1} + O(\frac{f(u)^{-n}}{n^{7/2}})$$

uniformly for $|u - 1| < \eta$. If $h(f(1), 1) \neq 0$ and X_n is defined as Equ.(3) for $y(x, u)$, then X_n is asymptotically normal with mean $(\mu + o(1))n$ and variance $(\sigma + o(1))n$ where μ and σ are some constants.

We can see that all the conditions hold for $t(x, u)$. Then, for the number of vertices of degree j, the following result is immediate.

Theorem 3. *Suppose j is an integer. Let X_n be the number of vertices of degree j in $\mathcal{T}_n$. Then, X_n is asymptotically normally distributed with mean value $(\mu_j + o(1))n$ and variance $(\sigma_j + o(1))n$, where μ_j and σ_j are some constants to every j.*

Following book [1], we will say that *almost every* (a.e.) graph in a random graph space $\mathcal{G}_n$ has a certain property Q if the probability $\Pr(Q)$ in $\mathcal{G}_n$ converges to 1 as n tends to infinity. Occasionally, we shall write *almost all* instead of almost every.

For the number of vertices of degree j, by Chebyshev inequality one can get that

$$\Pr\left[\left|X_n - E(X_n)\right| > n^{3/4}\right] \leq \frac{Var X_n}{n^{3/2}} \to 0 \text{ as } n \to \infty.$$

Therefore, $X_n = (\mu_j + o(1))n$ a.e. Then, an immediate consequence is the following.

Corollary 4. *For almost all trees in $\mathcal{T}_n^\Delta$, the number of vertex of degree j is $(\mu_j + o(1))n$.*

Now, we discuss the general Zagreb index. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. The *general Zagreb index* was introduced by Li *et al.* [10], where they call it *the zeroth order general Randić index*, and is defined to be the sum of powers of degree, i.e.,

$$D_\alpha = \sum_{u \in \mathcal{V}} d_u^\alpha,$$

where α is some real number, d_u is the degree of vertex u. Many results have been obtained for this variable. Particularly, if $\alpha = -1$, D_{-1} is called the *inverse degree*, and if $\alpha = 2$, D_2 is known as the *first Zagreb index* [6].

For a tree with n vertices, if $\alpha = 0$ then $D_0 = n$. So, we focus on the case $\alpha \neq 0$, and establish estimate to the value of D_α for almost all trees in $\mathcal{T}_n^\Delta$.

Since the degrees of the tree in $\mathcal{T}_n^\Delta$ are bounded by Δ, we can obtain that, for almost all trees,

$$D_\alpha = \sum_{j=1}^{\Delta} j^\alpha \cdot (\mu_j + o(1))n.$$

For convenience, set $d_\alpha = \sum_{j=1}^{\Delta} j^\alpha \cdot \mu_j$.

Corollary 5. *For almost all trees in $\mathcal{T}_n^\Delta$, the value of the general Zagreb index enjoys*

$$D_\alpha = (d_\alpha + o(1))n,$$

where d_α is a constant.

3 The number of edges of type (i, j)

We start this section by counting the number of edges of type (i, j). Without loss of generality, suppose $i \leq j$. Since there is only one tree with an edge of type $(1, 1)$, we always assume $j > 1$.

In this section, we still use the same notation as in Section 2. We also use X_n to denote the number of edges of type (i, j), which would not make any ambiguity. Split up P_n^Δ into Δ subsets according to the degrees of the roots, and let $a_k(x, u)$ (or a_k) be the generating function corresponding to each subset. Then, we have $a_1(x, u) = x$ and

$$x + a_2(x, u) + \cdots + a_\Delta(x, u) = p(x, u).$$

Firstly, we establish the functions system and use Lemma 1 to get equations for $a_k(x, u)$ and $p(x, u)$ in the form of Equ.(7). Analogous to Equ.(10), we can obtain equations below

$$a_2(x, u) = x \cdot p(x, u),$$

$$\cdots\cdots$$

$$a_i(x, u) = x \cdot \sum_{\ell_1 + \ell_2 = i-1} Z(S_{\ell_1}; p(x, u) - a_j) \cdot Z(S_{\ell_2}; a_j) u^{\ell_2},$$

$$\cdots\cdots$$

$$a_j(x, u) = x \cdot \sum_{m_1 + m_2 = j-1} Z(S_{m_1}; p(x, u) - a_i) \cdot Z(S_{m_2}; a_i) u^{m_2},$$

$$\cdots\cdots$$

$$a_\Delta(x, u) = x \cdot Z(S_{\Delta-1}; p(x, u)).$$

With respect to $a_i(x, u)$, the root of a tree is attached by $i - 1$ planted trees. Then, suppose there are ℓ_2 planted trees attached to the root that has degree j while the other ℓ_1 planted trees are not, for which, in the above equations the term $Z(S_{\ell_1}; p(x, u) - a_j) \cdot Z(S_{\ell_2}; a_j)u^{\ell_2}$ is to treat this case. Hence, the above functions system follows.

Moreover, we shall show that these functions satisfy all the conditions of Lemma 1. Since the others are easy to verify, we only show that Equ.(5) holds. Set $\mathbf{a}(x, u) = (a_2, \ldots, a_\Delta)$. In fact, by using Equ.(11), for $k \geq 2$ it is easy to get

$$
\mathbf{F}_{a_k}(x_0, \mathbf{a}(x_0, 1), 1) = \begin{pmatrix} x_0 \\ \cdots \\ x_0 \cdot Z(S_{j-2}; p(x_0, 1)) \\ \cdots \\ x_0 \cdot Z(S_{\Delta-2}; p(x_0, 1)) \end{pmatrix}.
$$

Recalling that $p^{(\Delta-1)}(x_0, 1) = p^{(\Delta-1)}(x_0) = 1$, one can readily see that

$$
\det(\mathbf{I} - \mathbf{F_a}(x_0, \mathbf{a}(x_0, 1), 1)) = 0.
$$

Moreover, we have $\mathbf{v}^T = (1, \ldots, 1)$. Then from Lemma 1, we have

$$
a_k(x, u) = g_k(x, u) - h_k(x, u)\sqrt{1 - \frac{x}{f(u)}} \tag{17}
$$

where $2 \leq k \leq \Delta$, and thus

$$
p(x, u) = g_0(x, u) - h_0(x, u)\sqrt{1 - \frac{x}{f(u)}}, \tag{18}
$$

where g_k, h_k, g_0, h_0 and $f(u)$ are as required in Lemma 1. Since $p(x, u)$ is the sum of a_k's, which is the function in $\mathbf{a}(x, u)$, x and u, by Lemma 1 we have $\frac{-f'(1)}{f(1)} = \frac{1}{x_0} \frac{\mathbf{v}^T(\mathbf{F}_u(x_0, \mathbf{a}_0, 1))}{\mathbf{v}^T(\mathbf{F}_x(x_0, \mathbf{a}_0, 1))}$ and $f(1) = x_0$. From the functions system of a_k's, we get that $\mathbf{F}_u(x_0, \mathbf{a}_0, 1)$ and $\mathbf{F}_x(x_0, \mathbf{a}_0, 1)$ are positive and thus $\frac{-f'(1)}{f(1)} > 0$.

For the rooted trees, we have

$$
r(x, u) = x \cdot \sum_{k=1}^{\Delta} Z(S_k; p(x, u)) + x \sum_{l_1+l_2=i} Z(S_{l_1}; p(x, u) - a_j) \cdot Z(S_{l_2}; a_j)(u^{l_2} - 1)
$$
$$
+ x \cdot \sum_{m_1+m_2=j} Z(S_{m_1}; p(x, u) - a_i) \cdot Z(S_{m_2}; a_i)(u^{m_2} - 1).
$$

Note that if we join two planted trees with roots of degree i and j, respectively, then the number of edges of type (i, j) in the new tree is counted by $a_i(x, u)a_j(x, u)u$. Therefore, it follows that if $i < j$, then

$$
t(x, u) = r(x, u) - \frac{1}{2}p(x, u)^2 + \frac{1}{2}p(x^2, u^2) + a_i a_j(1 - u),
$$

and if $i = j$, then

$$t(x, u) = r(x, u) - \frac{1}{2}p(x, u)^2 + \frac{1}{2}p(x^2, u^2) + \frac{1}{2}a_i^2(1 - u) - \frac{1}{2}a_i(x^2, u^2)(1 - u).$$

By using Equs.(18) and (17), we always obtain that $t(x, u)$ is in the form of

$$t(x, u) = \overline{g} - \overline{h}\sqrt{1 - \frac{x}{f(u)}}.$$

Surely, we can proceed to show that $\overline{h}(f(u), u) = 0$ around $u = 1$ as previous and obtain that the random variable X_n to the edges of type (i, j) is still asymptotically normal. But it is much involved in this case. Thus, we introduce the following lemma (see [8]), which can give us a weak result.

Lemma 6. *Suppose $y(x, u)$ has the form*

$$y(x, u) = g(x, u) - h(x, u)\sqrt{1 - \frac{x}{f(u)}}$$

where $g(x, u)$, $h(x, u)$ and $f(u)$ are analytic functions around $x = f(1)$ and $u = 1$ that satisfy $h(f(1), 1) = 0$, $h_x(f(1), 1) \neq 0$, $f(1) > 0$ and $f'(1) < 0$. Furthermore, $x = f(u)$ is the only singularity on the cycle $|x| = |f(u)|$ for u is close to 1. Suppose X_n is defined as Equ.(3) to $y(x, u)$. Then, $E(X_n) = (\mu + o(1))n$ and $Var(X_n) = (\sigma + o(1))n$, where $\mu = -f'(1)/f(1)$ and $\sigma = \mu^2 + \mu - f''(1)/f(1)$.

Remark 2: This result does not tell us that the limiting distribution is asymptotically normal. If $h(f(1), 1) \neq 0$, this lemma is trivial by Lemma 1, and if $h(f(u), u) = 0$, we can still get that the limiting distribution is normal.

For $t(x, u)$, since $t(x, 1) = t(x)$, we have that $\overline{h}(f(1), 1) = 0$ and $\overline{h}_x(f(1), 1) \neq 0$. Moreover, the other conditions in Lemma 6 immediately follow from Equs.(17) and (18). Then, we can establish the following theorem.

Theorem 7. *Let X_n be the number of edges of type (i, j) in $\mathcal{T}_n^\Delta$. Then,*

$$E(X_n) = (\mu_{ij} + o(1))n$$

and

$$Var(X_n) = (\sigma_{ij} + o(1))n$$

where μ_{ij} and σ_{ij} are some constants to every type (i, j).

Consequently, by Chebyshev inequality we have the following result.

Corollary 8. *For almost all trees in $\mathcal{T}_n^\Delta$, the number of edges of type (i, j) equals $(\mu_{ij} + o(1))n$.*

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Now, we can give estimate to the value of the general Randić index for trees in $\mathcal{T}_n^\Delta$. The *general Randić index* is defined as

$$R_\beta = \sum_{uv \in \mathcal{E}} (d_u d_v)^\beta,$$

where d_u, d_v are the degree of u, v, respectively. If $\beta = -1/2$, the index is called the classic *Randić index* [12]. If $\beta = 1$, R_1 is known as the *second Zagreb index* [6]. We refer the readers to [9] for a detailed survey. Moreover, for a tree with n vertices, if $\beta = 0$ then $R_0 = n - 1$. Thus, we suppose $\beta \neq 0$. We shall get the estimate of R_β for almost all trees.

By Corollary 8, for trees in $\mathcal{T}_n^\Delta$, we can obtain that

$$R_\beta = \sum_{i \leq j \leq \Delta} (ij)^\beta \cdot (\mu_{ij} + o(1))n \text{ a.e.}$$

Denote $\sum_{i \leq j \leq \Delta} (ij)^\beta \cdot \mu_{ij}$ by r_β. Then the following result is immediate.

Corollary 9. *For almost all trees in $\mathcal{T}_n^\Delta$, the general Randić index R_β equals $(r_\beta + o(1))n$, where r_β is a constant.*

4   Concluding remark

Although the general Zagreb index and the general Randić index for trees and general graphs have been studied extensively, there are very few results from the asymptotic point of view to study them. Almost all known results are about extremal values of the indices and trees or graphs that attain the extremal values. It is known (see [9]) that for trees the maximum of the first Zagreb index and the Randić index is, respectively, $n(n-1)$ (attained by the star S_n) and $\frac{n-3}{2} + \sqrt{2}$ (attained by the path P_n), while the minimum of them is, respectively, $4n - 6$ (attained by the path P_n) and $\sqrt{n-1}$ (attained by the star S_n). Our results show that for almost all trees of bounded maximum degree the values of the indices are linear in the order n of the trees.

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